Exercise 1. Show that p.v.$\dfrac{1}{x} \in \mathscr{S}'(\mathbb{R})$.

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} e^{-x} & \text{si } x > 0 \\ 0 & \text{si } x \leq 0. \end{cases}$$

Show that $f \in \mathscr{S}(\mathbb{R})$ and deduce that $e^x \notin \mathscr{S}'(\mathbb{R})$.

Exercise 3. Compute $\mathscr{F}(1)$ and $\mathscr{F}(\mathrm{sgn}(x))$ in $\mathscr{S}'(\mathbb{R})$, where

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{si } x > 0 \\ 0 & \text{si } x = 0 \\ -1 & \text{si } x < 0. \end{cases}$$

Exercise 4.

1. Show that for all $s > \dfrac{d}{2}$, we have the continuous injection

$$H^s(\mathbb{R}^d) \hookrightarrow C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).$$

In other words, show that for all $u \in H^s(\mathbb{R}^d)$, we have $u \in C^0(\mathbb{R}^d)$, and show that there exists a universal constant $C_s < \infty$ such that for all $u \in H^s(\mathbb{R}^d)$, we have

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq C_s \|u\|_{\mathrm{H}^s(\mathbb{R}^d)}.$$

2. Deduce that for $s > \dfrac{d}{2}$, the space $H^s(\mathbb{R}^d)$ is an algebra. In other words, there exists a universal constant $C_s' < \infty$ such that for all $u, v \in H^s(\mathbb{R}^d)$, we have

$$\|u\, v\|_{\mathrm{H}^s(\mathbb{R}^d)} \leq C_s' \|u\|_{\mathrm{H}^s(\mathbb{R}^d)} \|v\|_{\mathrm{H}^s(\mathbb{R}^d)}.$$

Hint: First prove for all $s \geq 0$ the elementary inequality

$$\left(1+|\xi|^2\right)^{\frac{s}{2}} \leq 2^s \Big(\left(1+|\xi-\eta|^2\right)^{\frac{s}{2}} + \left(1+|\eta|^2\right)^{\frac{s}{2}} \Big) \qquad \text{for all } \xi, \eta \in \mathbb{R}^d$$

and use Young's inequality for convolution.

Exercise 5 (Sobolev embedding theorem)**.** Assume that $0 \leq s < \dfrac{d}{2}$. We are going to prove Sobolev embedding $H^s(\mathbb{R}^d) \longhookrightarrow L^{s^*}(\mathbb{R}^d)$, where $s^* = \dfrac{2d}{d-2s}$.

1. Let (X, μ) be a σ-finite measured space. Show that for all $f \in L^p(X, \mu)$, the following formula holds:

$$\int_X |f|^p d\mu = p \int_0^\infty t^p \, \mu\left(X \cap \{x : |f(x)| > t\}\right) \frac{dt}{t}.$$

Hint: use Fubini's theorem*.

2. Take $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and make the decomposition $f = g + h$ and fix some $t > 0$. Show that

$$\mathscr{L}^d(\{|f| > t\}) \leq \mathscr{L}^d\left(\left\{|g| > \frac{t}{2}\right\}\right) + \mathscr{L}^d\left(\left\{|h| > \frac{t}{2}\right\}\right)$$

where $\mathscr{L}^d$ is the Lebesgue measure.

*Actually, the result is true for any measured space, but the proof is more involved.

3. For all $t > 0$, and let A_t be a constant to determine later, and define the functions

$$\widehat{g_t}(\xi) = \begin{cases} \widehat{f}(\xi) & \text{for all } \ |\xi| \leq A_t \\ 0 & \text{for all } \ |x| > A_t \end{cases} \qquad \text{and} \qquad \widehat{h}_t(\xi) = \begin{cases} 0 & \text{for all } \ |\xi| \leq A_t \\ \widehat{f}(\xi) & \text{for all } \ |x| > A_t. \end{cases}$$

Show that $f = g_t + h_t$, and assume furthermore that A_t is determined (for all $t > 0$) in such a way that $\left\{|g_t| > \dfrac{t}{2}\right\} = \varnothing$. Prove that

$$\|f\|^p_{\mathrm{L}^p(\mathbb{R}^d)} \leq 4p \int_0^{\infty} t^{p-3} \, \|h_t\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \, dt.$$

Hint: use Markov's inequality.

4. Assuming without loss of generality that $\|f\|_{\mathrm{H}^s(\mathbb{R}^d)} \leq 1$, show that there exists a universal constant $\Lambda = \Lambda(d,s)$ such that

$$\|g_t\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq \Lambda \, A_t^{\frac{d}{2}-s}.$$

Deduce a choice of A_t such that $\|g_t\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq \dfrac{t}{2}$.

5. Using the previous choice of A_t and the Plancherel formula, show that there exists a universal constant $C = C(d,s)$ such that

$$\int_{\mathbb{R}^d} |f(x)|^p dx \leq C \int_{\mathbb{R}^d} |\xi|^{\frac{d(p-2)}{p}} |\widehat{f}(\xi)|^2 d\xi$$

and conclude the proof of the theorem.